

Even and Odd Cycles Passing a Given Edge or a Vertex

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Abstract

In this paper we provide some sufficient conditions for the existence of an odd or even cycle that passing a given vertex or an edge in 2-connected or 2-edge connected graphs. We provide some similar conditions for the existence of an odd or even circuit that passing a given vertex or an edge in 2-edge connected graphs. We show that if G is a 2-connected k -regular graph, $k \geq 3$, then every edge of G is contained in an even cycle. We also prove that in a 2-edge connected graph, if a vertex has odd degree, then there is an even cycle containing this vertex.

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1 Introduction

Throughout this paper all graphs are simple with no loops and multiple edges. Let G be a graph with vertex set and edge set $V(G)$ and $E(G)$, respectively. If $v \in V(G)$, then $N(v)$ denotes the set of all neighbors of v and $d_G(v) = |N(v)|$ is called the *degree* of v . If every vertex of G has the same degree k , then G is called a *k-regular* graph.

A graph G is said to be *k-connected* if it has more than k vertices and remains connected whenever fewer than k vertices are removed, and is *k-edge connected* if it remains connected whenever fewer than k edges are removed.

A *walk* is a sequence of vertices and edges $v_0, e_1, v_1, \dots, e_k, v_k$, such that for $i = 1, \dots, k$, the edge e_i has endpoints v_{i-1} and v_i . A *trail* is a walk with no repeated edge. A *circuit* is a trail that its endpoints are the same. Two paths are *internally vertex disjoint* if they do not have any internal vertex in common. Let C be a cycle in G and $e = \{u, v\} \in E(G) \setminus E(C)$. If $\{u, v\} \subseteq V(C)$, then e is called a *chord* of C .

There are some results on the existence of cycles passing a given subset of vertices or edges in a graph. In 1960, Dirac proved that for every set of k vertices in a k -connected graph there exists a cycle that passes through all vertices of the set, see [2]. In 1977, Woodall proved that given any l disjoint edges in a $(2l - 2)$ -connected graph, $l \geq 2$, there is a cycle containing all of them, see [5]. In 1981, Bondy and Lovász showed that if S is a set of k vertices in a k -connected graph G , $k \geq 3$, then there exists an even cycle in G through every vertex of S , see [1]. In [3], Häggkvist and Thomassen showed that if L is a set of k independent edges in a graph G such that any two vertices incident with L are connected by $k + 1$ internally disjoint paths, then G has a cycle containing all edges of L . In this paper, we consider some conditions on 2-connected graphs eventuating existence of cycles with different parity that passes a given vertex or a given edge.

Here, we prove that if G is a 2-connected k -regular graph, $k \geq 3$, then every edge of G is contained in an even cycle. We also show that if G is a 2-edge connected graph and v is a vertex of odd degree in G , then v is contained in an even cycle. Also, we prove that if G is a 2-connected non-bipartite graph, then every edge of G is contained in an odd cycle. Finally, we show that if G is a 2-edge connected k -regular graph, $k \geq 3$, then every edge of G is contained in an even circuit.

2 Cycles in Graphs Passing a Given Vertex or an Edge

We start this section with the following theorem.

Theorem 1. *Let G be a 2-connected graph and C be an odd cycle in G . Then every $e \in E(G) \setminus E(C)$ is contained in an even cycle.*

Proof. We claim that there are two vertex disjoint paths starting from two end points of $e = \{u, v\}$ to two end points of an arbitrary edge $f \in E(C)$. We add a new vertex on both e and f . Clearly, G remains 2-connected. Therefore, there are two internally vertex disjoint paths between these two new vertices [4, p.161]. These paths without new edges are the desired paths and the claim is proved. Call these two paths P and Q . Suppose that $P = uu_2 \cdots u_n$ and $Q = vv_2 \cdots v_m$. Let u_i and v_j be the first vertices of P and Q in $V(P) \cap V(C)$ and $V(Q) \cap V(C)$, respectively. Consider two paths $P' = uu_2 \cdots u_i$ and $Q' = vv_2 \cdots v_j$. Now, we have three cases:

Case 1. $V(C) \cap \{u, v\} = \emptyset$. Suppose that P and Q are two paths from u_i to v_j such that $V(P) \cup V(Q) = V(C)$ and $V(P) \cap V(Q) = \{u_i, v_j\}$. Since C is an odd cycle the parity of P and Q are different. Hence one of the cycles $eP'PQ'$ and $eP'QQ'$ is an even cycle, as desired.

Case 2. $V(C) \cap \{u, v\} = \{v\}$. Since G is 2-connected there exists a shortest path from u to C not containing v . We call this path by S and let $V(C) \cap V(S) = \{s\}$. Suppose that P and Q are two paths from s to v , $V(P) \cup V(Q) = V(C)$ and $V(P) \cap V(Q) = \{s, v\}$. Since C is an odd cycle the parity of P and Q are different. Hence one of the cycles eSP and eSQ is even.

Case 3. The edge e is a chord of C . Suppose that P and Q are two paths from u to v such that $V(P) \cup V(Q) = V(C)$ and $V(P) \cap V(Q) = \{u, v\}$. Since C is an odd cycle so one of the cycles eP and eQ is even. The proof is complete. \square

Now, we have the following corollary.

Corollary 1. *Let G be a 2-connected graph. If removing of every edge of G does not make the graph bipartite, then every edge of G is contained in an even cycle.*

Theorem 2. *Let G be a 2-connected non-bipartite graph. Then every edge of G is contained in an odd cycle.*

Proof. Since G is non-bipartite so it has at least one odd cycle. Let C be an odd cycle and $e \in E(G)$. If $e \in E(C)$, then we are done. If $e \notin E(C)$, then the proof is similar to Theorem 1 and e is contained in an odd cycle. The proof is complete. \square

Remark 1. The 2-connectivity condition in Theorem 2 is not superfluous, as shown in Figure 1. The graph in Figure 1 is non-bipartite but this graph is not 2-connected. The edge e is not contained in an odd cycle.

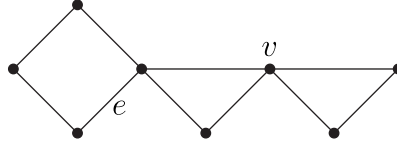


Figure 1. The edge e is not contained in an odd cycle. The vertex v is not contained in an even cycle.

By Theorem 1, we have the following result.

Theorem 3. *Let G be a 2-connected graph and $k \geq 3$ be a positive integer. If all vertices of G have degree divisible by k , then every edge of G is contained in an even cycle.*

Proof. First assume that G is bipartite. Since G is 2-connected so every edge of G is contained in an even cycle [4, p.162]. Next, suppose that G contains an odd cycle, say C . Let $e = \{u, v\} \in E(G)$. If there exists an odd cycle not containing e , then by Theorem 1, e is contained in an even cycle. Thus assume that every odd cycle of G contains e . Obviously, $H = G \setminus e$ is a bipartite graph. Since $e \in E(C)$, there is a path of even length between u and v in $H = (X, Y)$. Clearly, u and v are in the same part of H , say X . Since $d_H(u) \equiv d_H(v) \equiv -1 \pmod{k}$, so the sum of vertex degrees in X is $-2 \pmod{k}$, but the sum of vertex degrees in Y is $0 \pmod{k}$, a contradiction. The proof is complete. \square

Corollary 2. *Let G be a 2-connected k -regular graph, $k \geq 3$. Then every edge of G is contained in an even cycle.*

Remark 2. The divisibility condition in Theorem 3 is required. To see this, look at the Figure 2. The edge e of the following 2-connected graph is not contained in an even cycle.

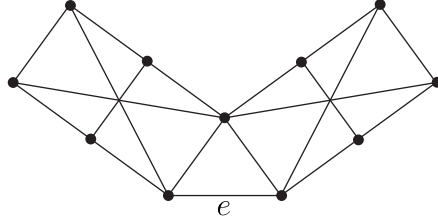


Figure 2. The edge e is not contained in an even cycle.

Remark 3. The 2-connectivity condition in Theorem 3 is not superfluous. To see this, look at the Figure 3. In this figure an edge between a component and a vertex means that all vertices of the component are adjacent to the vertex. Also an edge between two components means all vertices of these components are adjacent. The graph in Figure 3 is k -regular but this graph is not 2-connected. Obviously, the edge e is not contained in an even cycle.

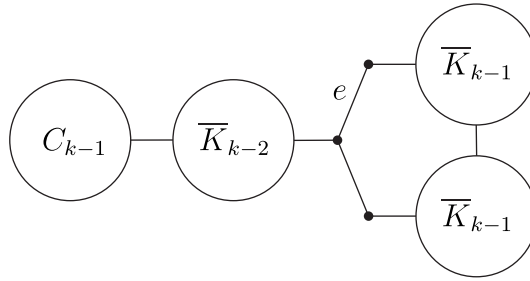


Figure 3. A connected graph containing an edge which is not contained in an even cycle.

Theorem 4. *Let G be a 2-connected graph, $v \in V(G)$ and $d(v) \geq 3$. Then v is contained in an even cycle.*

Proof. Since G is 2-connected, v is contained in a cycle. Call this cycle by C . If C is an even cycle, then we are done. Thus assume that C is an odd cycle. Since $d(v) \geq 3$, v is incident with another edge e , not contained in C . Since $e \notin E(C)$, so by Theorem 1, e is contained in an even cycle. \square

Remark 4. The 2-connectivity condition in Theorem 4 is not superfluous. To see this, look at the Figure 1.

Theorem 5. *Let G be a 2-edge connected graph and $v \in V(G)$. If $d(v)$ is odd, then there is an even cycle containing v .*

Proof. We claim that $G \setminus v$ has a connected component H such that $|V(H) \cap N(v)| \geq 3$. Since G is 2-edge connected, in each connected component of $G \setminus v$ there are at least two neighbors of v . Now, assume that for each connected component of G , say H , $|V(H) \cap N(v)| = 2$. This implies that $d(v)$ is even, a contradiction. Thus, there exists a connected component of $G \setminus v$, say H , such that $|V(H) \cap N(v)| \geq 3$.

Let $\{x, y, z\} \subseteq V(H) \cap N(v)$. With no loss of generality assume that $\{x, y\}$ has minimum distance among all pairs of $\{x, y, z\}$ in $G \setminus v$. Suppose that M is a path of minimum length between x and y . Since H is connected, there exists a shortest path, say N , between z and M . Let $w \in V(N) \cap V(M)$ and P and Q are two paths such that $V(P) \cup V(Q) = V(M)$, $V(P) \cap V(Q) = \{w\}$, $x \in V(P)$ and $y \in V(Q)$. We denote the edges vx , vy and vz by e_{vx} , e_{vy} and e_{vz} , respectively. If two cycles $e_{vx}Me_{vy}$ and $e_{vy}QNe_{vz}$ are odd, then $l(e_{vx}PNe_{vz}) = l(e_{vx}Me_{vy}) + l(e_{vy}QNe_{vz}) - 2l(e_{vy}Q)$ which is even, where $l(R)$ denotes the length of R , as desired. \square

Remark 5. Being odd for $d(v)$ is required in Theorem 5. The graph in Figure 4 is 2-edge connected but the degree of v is even and v is contained in no even cycle.

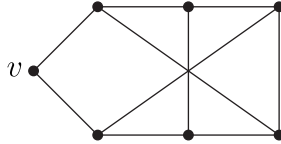


Figure 4. The vertex v in the graph has even degree and there is no even cycle containing v .

Now, we present some results on the existence of even or odd circuit passing a given edge or vertex.

Theorem 6. *Let G be a 2-edge connected graph and C be an odd cycle in G . Then every $e \in E(G) \setminus E(C)$ is contained in an even circuit.*

Proof. The proof is similar to Theorem 1.

Using the method applied in the proof of Theorem 3, one can prove the following result.

Theorem 7. *Let G be a 2-edge connected graph and $k \geq 3$ be a positive integer. If all vertices of G have degree divisible by k , then every edge of G is contained in an even circuit.*

Remark 6. The divisibility condition in Theorem 7 is required. To see this, look at the Figure 2.

If one applies the idea of the proof of Theorems 2 and 4, then the following results hold.

Theorem 8. *Let G be a 2-edge connected graph and $v \in V(G)$. If $d(v) \geq 3$, then v is contained in an even circuit.*

We close this paper with the following result.

Theorem 9. *Let G be a 2-edge connected non-bipartite graph. Then every edge of G is contained in an odd circuit.*

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